WPI Mathematical Sciences Ph.D. General Comprehensive Exam

MA 540 Probability and Mathematical Statistics-I

August, 2017

Note: Please make sure to write down your thinking process in bullet points even if you cannot solve the problems perfectly.

1. Suppose $X_i \overset{iid}{\sim} U(a,b),\ i = 1, \ldots, n$, where $0 < a < b < \infty$. The geometric mean of the X_i is $G = [\prod_{i=1}^{n} X_i]^{(1/n)}$.
   (a) Find the expected value of G as a function of a, b, and n.
   (b) Does G converge in probability? With probability 1? If it does converge, obtain its limit. If it does not converge, prove that it does not.
2. Let X be a random variable with $EX^2 < \infty$ and a pdf symmetric about 0, and let $Y = |X|$. Are X and Y uncorrelated? Are they independent? Prove your results or give counterexamples.
3. Suppose $X \geq 0$ has pdf f, that $p > 0$, and that $EX^p < \infty$. Show that $EX^p = \int_0^\infty px^{p-1}P(X > x)dx$.
4. Let $X_1, X_2, \ldots$ be a sequence of iid continuous random variables. We say a record occurs at time n if $X_n > \max\{X_1, \ldots, X_{n-1}\}$.
   (a) Show that the probability a record occurs at time n equals $1/n$.
   (b) Let N be the number of records that occur by time n Find $E(N)$.
5. Suppose a fair coin is flipped until H appears in two successive flips. Let X denote the total number of flips for this to happen. Give a formula for the probability mass function of X.
6. Let X be a random variable with probability density (or mass) function $f(x)$, and assume that $E(e^X) < \infty$. Construct $g(x) = ke^x f(x)$, a probability density (or mass) function. Write down a form for k. Determine $g(x)$ if (a) $f(x)$ is Normal(μ, σ^2), and (b) $f(x)$ is Binomial(n, p).